Metapopulation and Coupled - Logistic Maps

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Outline

Vahia (2011) theorized that the Indus Valley Civilization declined due to the failure of society to solve increasing needs of the civilization because of delay in arrival of new technologies.
In a personal discussion, he elaborated on how the migration and settlement of species is governed by Non Linear Dynamics

Uncountable games of Microsoft Age of Empires led to an intuitive belief that the game results have a chaotic nature. The army count time series shows in phase oscillations leading to extinction for one party and saturation for the other.

Motivation
Age of Empires III

Plan A: Food and Area

Introduction

We came to a consensus that other than population, for the evolution of a single city, we should try at least two other variables namely food and area. Their definitions evolved through the course of our project as we gained more insight into how they're governed. The three variables of a single city are formally defined as:

Population

Representative of the number of people living in a city

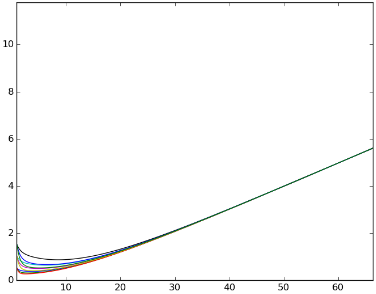
Food

Represents the amount of food stored in excess for a unit time, negative value is indicative of a scarcity.

Area

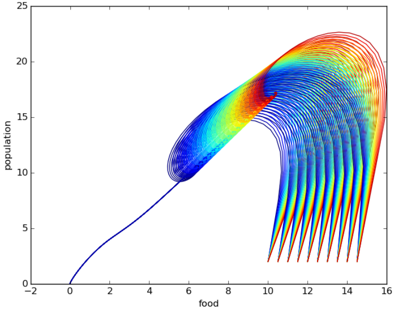
Represents the area claimed by civilization, to be used for housing and cultivation both in proportions.

$$\dot{A} = 0.01P - A$$
$$\dot{F} = (0.1P - F)(A - 0.02P)$$
$$\dot{P} = 0.5P - 0.5(0.1P - F)$$



Plan A: Food and Area
Model II

$$\dot{A} = 100Ae^{-A}(P - A)$$
$$\dot{F} = 0.7A - P$$
$$\dot{P} = 2.75\frac{F^2}{P} - P$$



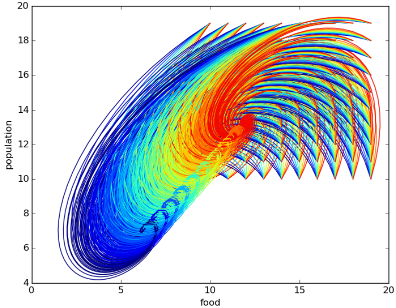
Plan A: Food and Area

Model III

$$\dot{A} = \begin{cases} \max(\dot{P}, 0) & P > A \\ 0 & otherwise \end{cases}$$

$$\dot{F} = 0.7A - P$$

$$\dot{P} = 1.125F - P$$



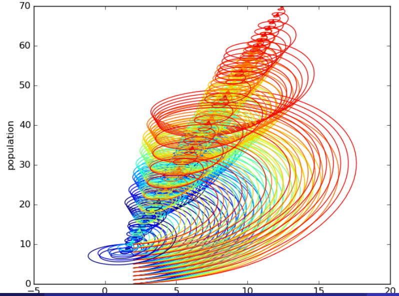
Plan A: Food and Area

Model IV

$$\dot{A} = \begin{cases} P - A & P > A \\ \max(\dot{P}, 0) & 0.3A < P < A \\ 0 & otherwise \end{cases}$$

$$\dot{F} = 0.7A - P$$

$$\dot{P} = 1.125F - 0.2P$$



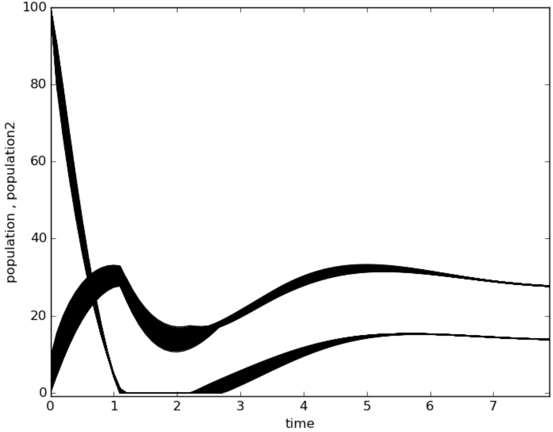
$$\dot{A} = \begin{cases} P - A & P > A \\ \max(\dot{P}, 0) & 0.3A < P < A \\ 0 & otherwise \end{cases}$$

$$\dot{F} = 0.7A - P$$

$$\dot{P} = 1.125F - 0.2P \pm M$$

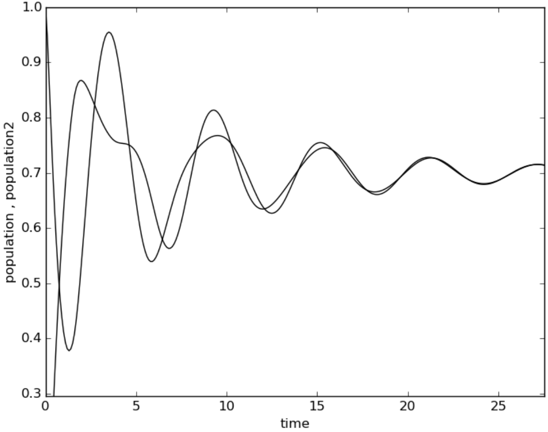
$$\dot{M} = P'(\frac{F}{P + 0.1} - \frac{F'}{P' + 0.1})$$

Plan A: Food and Area
Model V:Multiple Cities



Plan A: Food and Area

Model V:Multiple Cities



Where is the Chaos?

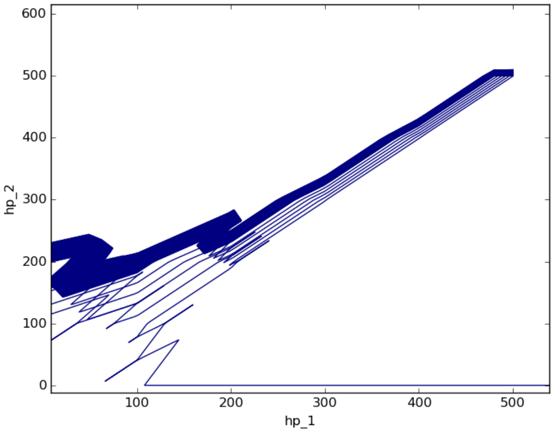
Hidden Non-Linearity

This model justified our claim that the possibility of migration stabilizes the system, in the sense that the population is saved from extinction as they have an option to migrate to locations with better opportunities. But our initial motivation was an intuition towards chaos as seen in AoE. It was still missing.

A careful examination of the game dynamics and how battles are fought led to the observation that two soldiers are not equivalent to a single soldier with twice the health and attack power. This is the non-linearity in the system which leads to possible chaos.

$1 + 1 \neq 2$

Solution



After failing to model a population of populations in a satisfactory manner, we came across Levins (1969) who coined the term **metapopulation** for the kind of systems we were aiming to study, with the variables being the carrying capacity of the patches of lands, migration costs and population. Although, his equation just took account of evolution of the fraction of such patches populated in terms of settlement and extinction coefficients.

$$\frac{dN}{dt} = cN(1 - N) - eN$$

Now we had the term we needed to search with, and we came upon a beautiful analysis on Coupled Logistic Maps in Llyods (1994). It demonstrated our idea of stabilization due to migration and coupling, along with rich properties arising out of logistic maps.

Logistic Maps

Logistic Maps are recurrence relations of degree 2 often used as an example of how chaotic behaviour can arise from very simple non-linear dynamical equations.
It was popularised by biologist Robert May, who gave a discrete demographic model analogous to the logistic equation.
Mathematically, it is given by:

$$x_{n+1} = \mu x_n(1 - x_n)$$

Where x_n is between 0 and 1 and 'μ' is our parameter of interest in the domain [0, 4]

Coupled Logistic Maps

Alun.L.Lloyd

In his paper titled : *'The Coupled Logistic Map: A Simple Model for the Effects of Spatial Heterogeneity on Population Dynamics'* Lloyd describes a simple model consisting of two diffusively coupled logistic maps and used it to examine the effects of spatial heterogeneity on population dynamics.

In his paper, he studies the simplest biologically realistic model, which (in terms of non-dimensional variables) takes the form:

$$x_{n+1} = (1-\alpha)f(x_n) + \alpha f(y_n)$$

$$y_{n+1} = (1-\alpha)f(y_n) + \alpha f(x_n)$$

Where f(x) takes the standard form: $f(x) = \mu x(1-x)$

Coupled Logistic Maps

In the previous equations coupling is linear in nature, but many times the coupling can be bi-linear in nature. For example:

$$x_{n+1} = f(x_n) \pm \alpha f(x_n) f(y_n)$$

$$y_{n+1} = f(y_n) \pm \alpha f(x_n) f(y_n)$$

From now onward we will refer to the linear model as **1** and the bilinear model as **2**.
The behaviour of single logistic maps are well understood, so the method we use to analyse coupled maps is to fix μ and vary α.
In this way, we can study the behaviour of coupled maps whose individual behaviour is well known.
A single logistic has limited dynamics but a coupled logistic map can rich dynamic nature.

Quasiperiodic Orbits

On converting our logistic map to polar coordinates, we get:

$$r_{n+1} = \lambda r_n - r_n^3$$

$$\theta_{n+1} = \theta_n + \alpha$$

Clearly, we get an invariant closed circle at $r = \sqrt{\lambda - 1}$ and the dynamics on the circle are given by $\theta_{n+1} = \theta_n + \alpha$. If α is a rational multiple of 2π and $\alpha =$ p/q then the orbit will be periodic with period q. If α is an irrational multiple of 2π then the orbit will not close up. This is called a quasiperiodic orbit.
Quasiperiodic orbits show properties of both periodic and aperiodic orbits.

Lyapunov Exponent

The Lyapunov exponents λ_1 and λ_2 can be used to distinguish between chaotic, quasiperiodic, periodic and fixed point behaviour. These two exponents measure the long term average rates of divergence or convergence of nearby orbits in this two dimensional system.
With no loss of generality, assuming λ_1 to be larger,

- If λ_1 is positive, then nearby orbits diverge, there is sensitive dependence on initial conditions and hence chaos.
- If λ_1 is zero, the motion is quasiperiodic and the attractor is a torus, in our system this means a closed curve, and the orbit never closes; or it can be a stable cycle (if the orbit closes)
- If λ_1 is negative then we have a periodic orbit, a fixed point being a particular example of this.

Lyapunov exponents - General ideas

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- An intersting interpretation of Lyapunov exponents is the rate at which information is *lost or added* to the system.
- Is calculating Lyapunov exponents easy?
- No. We are limited because of exponential divergence and round-off errors.

> In theory, there is no difference between theory and practice. But, in practice, there is.
>
> — Richard M. Nixon

Analytic method for Lyapunov exponents

We consider a fiducial trajectory plus a tangent space approach.[1]

Algorithm

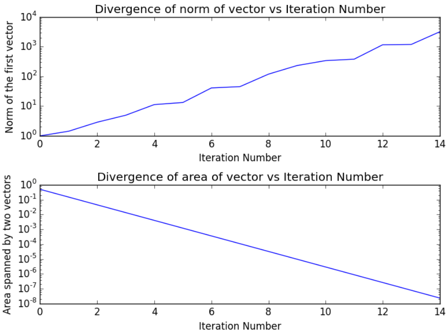
- Generate a *fiducial trajectory for a given set of initial conditions*
- Consider an orthonormal set of unit vectors which evolve according to the Jacobian of the system.
- However, each vector will tend to diverge along the direction of highest local growth. To circumvent this issue, we apply repeated Gram - Schmidt Orthonormalization on the vector frame after a certain number of iterations.
- The rate of growth of norm of the first vector, the area and so on give us the Lyapunov exponents.

[1] Benettin et al (1980)

Example - Henon Map

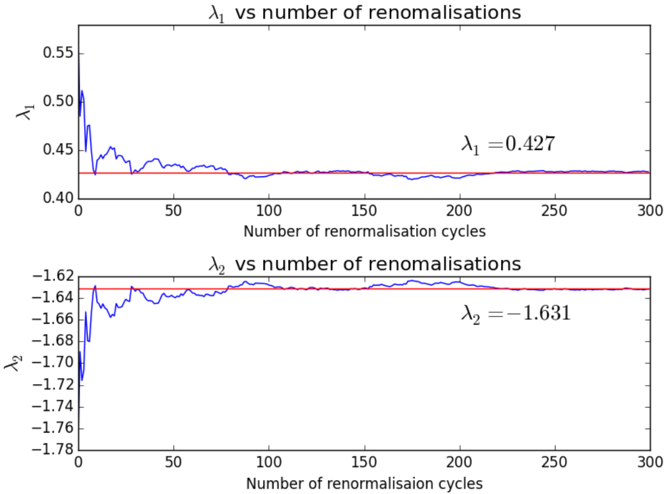
We consider the Henon map to illustrate the method. ($a = 1.4, b = 0.3$)

$$x_{n+1} = 1 - ax_n^2 + y_n$$
$$y_{n+1} = by_n$$



Example - Henon Map

The renormalisation was performed after $M = 15$ steps. A total of 300 such cycles were performed.



Attractor Reconstruction

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- Attractor reconstruction refers to methods for inference of **geometrical and topological information about a dynamical attractor only from observations.**
- Given the time series $x(t)$, where the system itself could have other independent dynamical variables, it is possible to reconstruct the phase space?
- Suprisingly enough, the answer turns out in the affirmative
- Given the time series $x(t)$, an m-dimensional phase portrait is reconstructed with delay coordinates. A point on the attractor is given by $x(t), x(t+\tau).....x(t+(m-1)\tau)$ where τ is the almost arbitrarily chosen delay time.

Embedding and Takens' Theorem

- **Embedding** The image of the n-manifold is completely unfolded in the larger space. In particular, no two points in the n-dimensional manifold map to the same point in the larger space.

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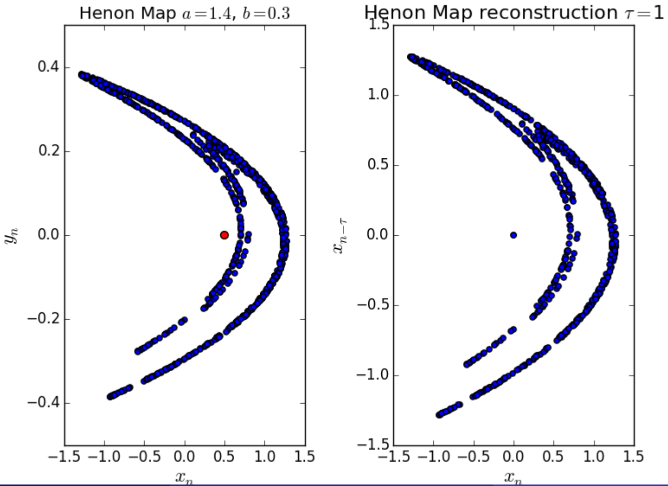
- A delay embedding theorem uses an observation function $(x(t))$ to construct the embedding function.
- **Takens' theorem** : The time-delayed versions $[x(t), x(t-\tau), x(t-2\tau), \cdots, x(t-2n\tau)]$ of one generic signal would suffice to embed the n-dimensional manifold. There are some technical assumptions that must be satisfied about the actual system.

Attractor Reconstruction

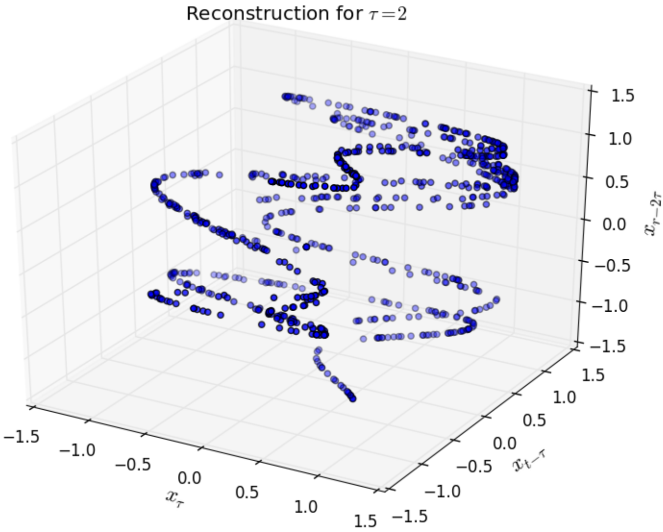
In Action

Here is phase space of the Henon system reconstructed using $\tau = 1$.
Except for a change of scale, there is a striking similarity between the plots.



Attractor Reconstruction

What if we choose the wrong τ?



Lyapunov Exponents from an Experimental Time Series

We present a method conceived by Wolf et al.We assume that the phase space has been reconstructed by the method discussed.

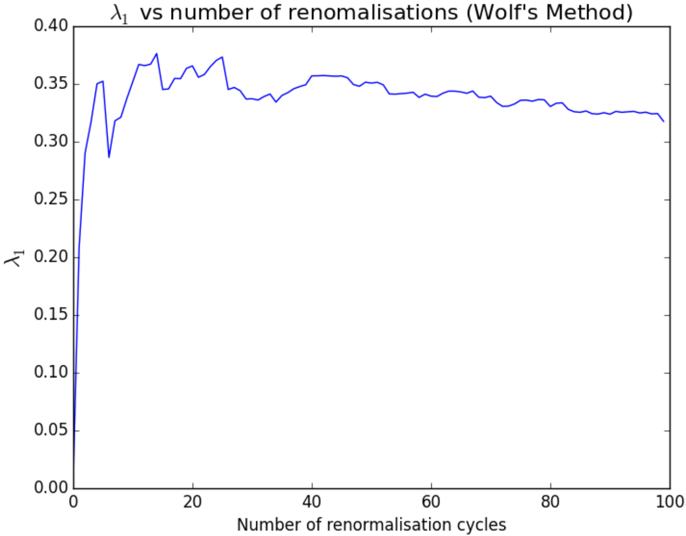
Wolf's Method

- We locate the nearest neighbor (in the Euclidean sense) to the initial point $(x(t_0).....x(t_0 + (m-1)\tau)$ and denote the distance between these two points $L(t_0)$.
- At a later time t_1, the initial length will have evolved to length $L'(t_1)$.
- This procedure is repeated until the fiducial trajectory has traversed the entire data file, at which point we estimate

$$\lambda_1 = \frac{1}{t_M - t_0} \sum_{k=1}^{M} \log_2 \frac{L'(t_k)}{L(t_{k-1})}$$

where M is the total number of replacement steps.

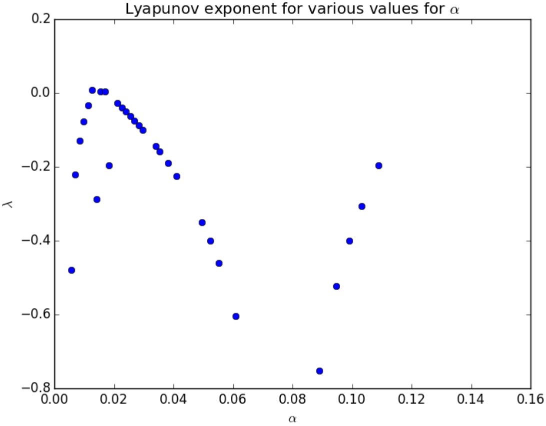
Wolf's Method
In Action



Lloyd Model

Back to Business

Using the methods developed, the the largest Lyapunov exponent for the coupled logistic map was calculated for various values of α.



How can we analyze the coupled equations?

- One way of summarizing the behaviour of a system like (1) or (2) is to make a two-dimensional bifurcation diagram consisting of a grid of points in the (μ,α) plane which we set to different colours according to the behaviour seen at each pair of parameter values. Such diagrams exhibit beautiful fractal structure and self-similarity.
- However,the behaviour of the system not only depends on the two parameters but also on the co-ordinates of the initial point, so these diagrams really should be four-dimensional as the set (μ,α,x,y) is needed.

How can we analyze the coupled equations?

- The first is to use many different initial conditions for a given μ and α ,These are usually chosen at random. All of the attractors should be seen if enough initial points are chosen. However, if we are near to an a value where an attractor is created or destroyed (call this α^0), only a small set of initial conditions may tend to the attractor in which we are interested.
- However, we must always be aware that it is possible for attractors to be created and destroyed during small changes in parameter values.
- And for attractors having very small basins of attraction,It is very easy to miss a lot of behaviour in a numerical study.

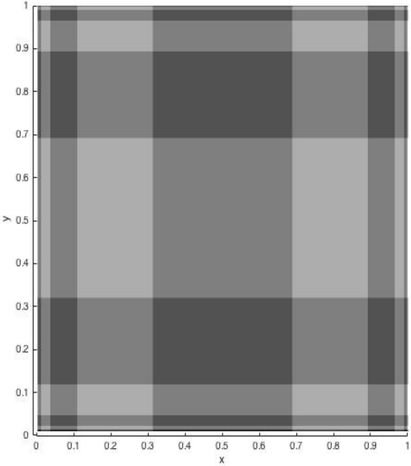
Now the question arises how to plot these graphs?

- Firstly we take 400*400 Initial values and will iterate them.
- Then we will calculate phase between them
- Finally we're going to plot then according to their phase difference. For a phase difference of zero, the point is given gray color,and will represent in-phase solution and negative and positive out of phase solutions are plotted by white and black respectively

Phase plots for μ=3.2

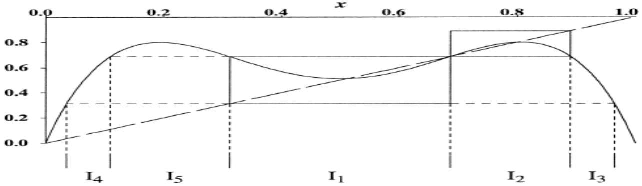
α=0

- While Plotting the graphs, my graphs looks like this for α =0



Phase plots for μ=3.2

- The interval I1 is invariant under f^2 and all points in its interior are attracted to the stable fixed point inside I1, similarly the interval I2 is invariant and all points in its interior are attracted to its stable fixed point. The intervals I3 and I4 map onto I1 under f^2, and interval I5 maps onto I2. We can continue this process, decomposing the unit interval into a collection of open intervals. Points on the boundaries of the intervals get mapped to the unstable fixed point



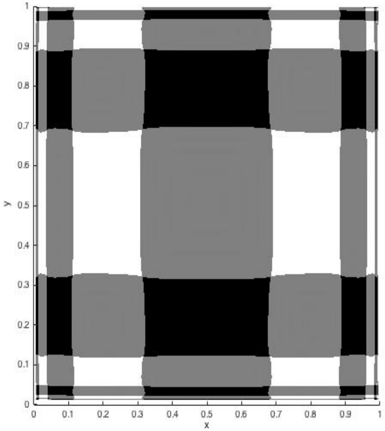
Phase plots for μ=3.2

- Thus when we couple two such maps with α=0, the basins of attraction are open rectangles formed by a cross product of the basins of attraction of the uncoupled maps.
- As α increases, the basin of attraction of the in phase solution grows and that of the out of phase solution shrinks.This can seen quite well in the next few slides

Phase plots for μ=3.2

α=.01

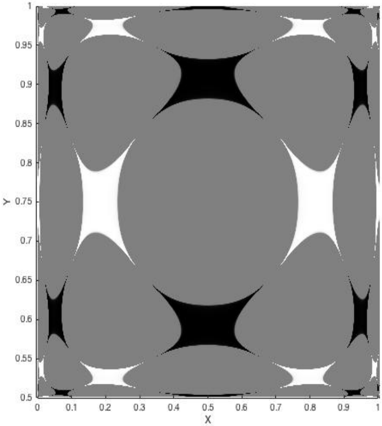
- Phase plot forα=.01



Phase plots for μ=3.2

α=.058

- Phase plot forα=.058. At this point, graph clearly shows the solutions tend to the inphase solutions.



Phase plots for μ=3.2

What happens if we increase more?

- As we increase α, the in phase solution remains stable but the out-of-phase solution loses its stability near α=0.058. This change occurs by a pitchfork bifurcation of the second iterate of the coupled map, as two unstable points collide with a stable point leaving a single unstable point.

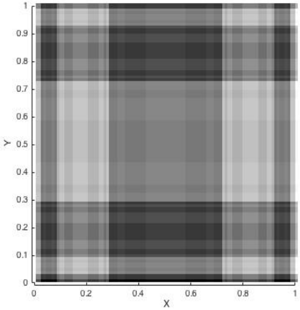
Phase plots for μ=3.5

- For this parameter value, the logistic map has α stable period-4 orbit, an unstable period-2 orbit, and two unstable fixed points. For α=0 there are four period-4 attractors, corresponding to x and y undergoing period-4 oscillations with four phase differences.
- The structure of the basins of attraction is explained in the same way as for the μ=3.2 case, except that one looks at f^4 instead of f^2

Phase plots for μ=3.5

α=0

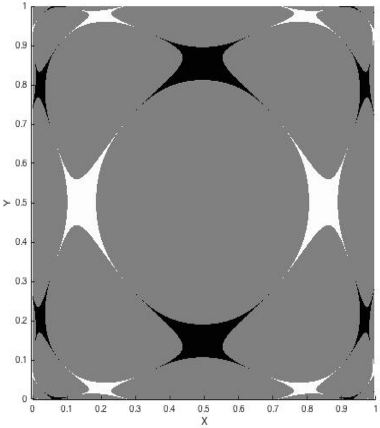
- This structure is quite similar to the case of μ=3.2 but with more number of squares



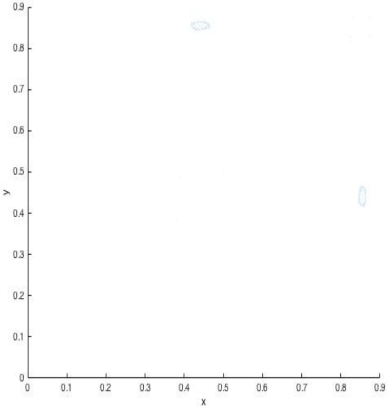
Phase plots for μ=3.5

α=0.12

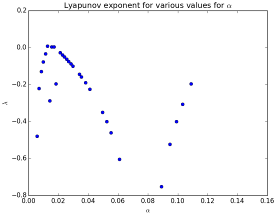
- Clearly the out of phase solutions are diminishing as we increase α



- As α is increased past about 0.0123, quasiperiodic behaviour is observed as two tori.



- The symmetric period-4 orbit loses stability when α is just below 0.0364, in a pitchfork bifurcation. The period-2 orbit loses stability by a pitchfork bifurcation for α just above 0.122, leaving the in-phase (diagonal) period-4 solution globally stable
- Figure is a plot of the largest Lyapunov exponent for each attractor seen as α increases from 0 to 0.15. The region for which quasiperiodic behaviour occurs is clearly seen as α range of α values for which this exponent is zero



Phase plots for μ=3.7

- The logistic map exhibits one band chaos for this parameter value. When the maps are coupled with α=0 we see just one attractor, which is square and symmetric about the diagonal. As the coupling is increased the attractor undergoes a complicated sequence of changes. Many intervals of a values give periodic behaviour, as is seen in the single logistic map beyond the onset of chaos. As a result of this, a numerical study of the system will tend to miss a lot of periodic behaviour.
- As a is increased between 0.0130 and 0.0135, the single block chaotic attractor suddenly changes into a two block attractor. A qualitative change in the structure of the chaotic attractors has occurred, such a change is known as a crisis. This type of crisis is called an interior crisis.

Figure: Phase Space Plots for μ=3.7 and α=.012

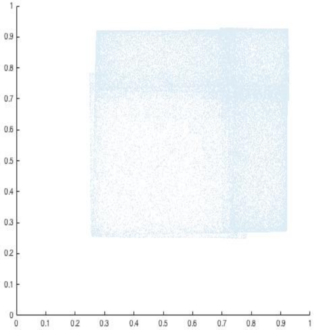
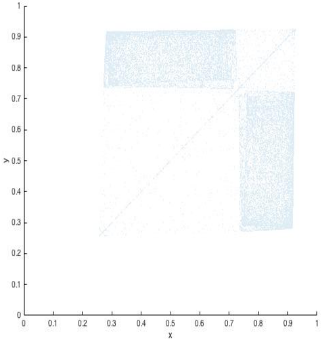
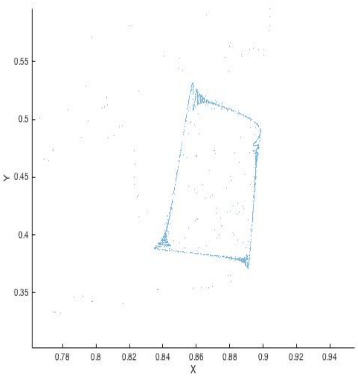


Figure: Phase Space Plots for μ=3.7 and α=.0135



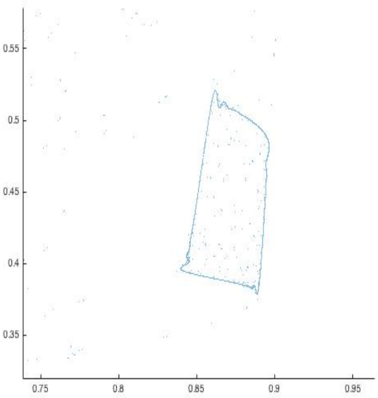
- Now we see some behaviour in between and will analyse α values between .55 and .70 to see what happens there.For α=.0574, the zoomed in tori looks like-



Plots for μ=3.7

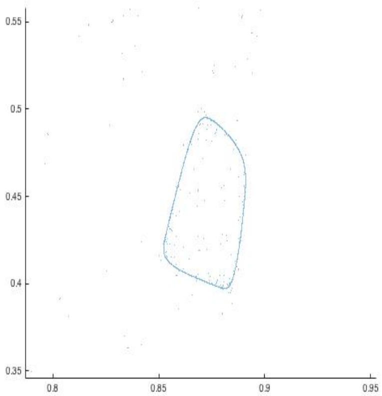
Analysis

- Plot for α=.0604



Plots for μ=3.7

- Plot for α=.0674



- Basically what we can see in these graphs that as the coupling is increased, the waves become straightened and the motion becomes quasiperiodic.
- Hence both of μ values, μ=3.5 and μ=3.7 show quasiperiodicity for some values of α.
- If you repeated these calculations for other μ values for which the logistic map shows 2^n band chaos. Similar bifurcation sequences are observed, with chaos going to quasiperiodicity, then to periodic behaviour. The in phase attractor is always favoured by large values of the coupling, with both x and y behaving like iterates of the single logistic map at the same μ value.

How is this useful?

- The coupled logistic map exhibits a much wider range of dynamic behaviour than the single logistic map, much of which may be important to the study of population dynamics.

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- Crisis behaviour may be more dramatic, with sudden changes in dynamic behaviour occurring for small changes in the parameters controlling the nature of the dynamics.

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- Crisis behaviour may be more dramatic, with sudden changes in dynamic behaviour occurring for small changes in the parameters controlling the nature of the dynamics.
- Finally this quote seems to come true

> You must have chaos within you to give birth to a dancing star.

Brief Review of the Model

- In our class we studied the logistic map, a simple 1-dimensional non-invertible map that exhibited chaos.
  $x_{n+1}=r\, x_n\, (1-x_n)$

- Now we introduced the **coupled-logistic map**, to reflect the **spatial heterogeneity** and consequently, the **possibility of migration** that is usually observed.
  $x_{n+1}=(1-D)\, f(x_n)+D\, f(y_n)$
  $y_{n+1}=D\, f(x_n)+(1-D)\, f(y_n)$
  where $f(x)=r\, x\, (1-x)$

The Question

Can Migration Stabilize Local Population Dynamics?

- At first the question seemed quite straightforward; except we needed to know what 'stabilisation' meant in this context.
- Does it mean that the system is non-chaotic? Or does it mean that the system, which models population and migration, ensures that the now 'stabilised' population doesn't go to extinction?
- Turns out they're not very different after all.
- But first we present results[2] of a stability analysis of the fixed points and 2- periodic orbits of this coupled system.

[2] A detailed proof can be found in *Does Migration Stabilise Local Population Dynamics? Analysis of a Discrete Population Model* by Gyllenberg et. al. For reasons that it is indeed mechanical and repetitive, it is not shown here.

Stability Analysis- Approach

- We are interested in the parameter space such that $0 \leq r \leq 4$ and $0 \leq D \leq 0.5$
- $(x, y) \in I^2 := [0,1] \times [0,1]$
- We are interested in the two 2-periodic orbits[3]

  In Phase

  (x,x): $x=(r+1)^{1/2}[(r+1)^{1/2} \pm (r-3)^{1/2}]/2r$

  Out of Phase

  $(\frac{u+v+1}{2}, \frac{u-v+1}{2})$: $u=\frac{1}{rg}$, $v=\pm \frac{(r^2-2rg^2-2g-1)^{1/2}}{rg}$

[3] For ease of solving, in the calculations of the second iterate and the Jacobian, there was a simple change of variables, u=x+y-1, v=x-y, g=1-2D

Stability Analysis-Results

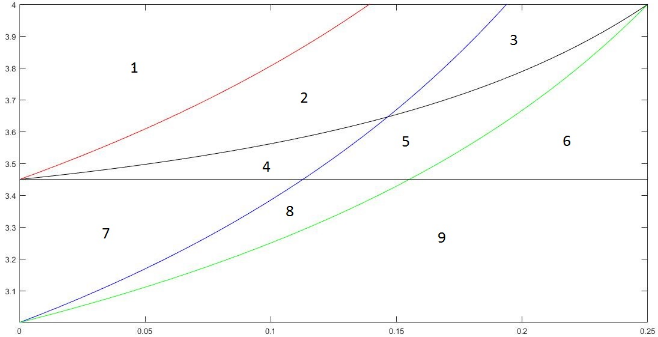
The **in phase orbit** exists only for r>3 and it is

- Stable for: $3 < \mathrm{r} < 1+\sqrt{6}$
- Saddle for: $0 < \mathrm{D} < 0.25$, $1+\sqrt{6} < \mathrm{r} < 1+(5+g^{-2})^{1/2}$
- Unstable for: $\mathrm{r} > 1+(5+g^{-2})^{1/2}$

The **out-of-phase orbit** exists in I^2 if
$0 < \mathrm{D} < 0.25$ and $\mathrm{r} > 2+1/\mathrm{g}$

- Stable for: $3g^{-2}+1 < (r-1)^2 < 3/\mathrm{g} + 2g^{-2} + 1$
- Saddle for: $1+1/g^2 < (r-1)^2 < 3g^{-2}+1$
- Unstable for: $(r-1)^2 > 3/\mathrm{g} + 2g^{-2} + 1$

Figure: Stability regions of the in-phase and out-of-phase 2 periodic orbits. Red: $3/g + 2g^{-2} + 1$ Blue: $3g^{-2} + 1$ Green: $2 + 1/g$



Focus on:

- Regions 7,8,9 below the horizontal line $r = 1+\sqrt{6}$
- Regions 7,4,2 between the red and blue curves.

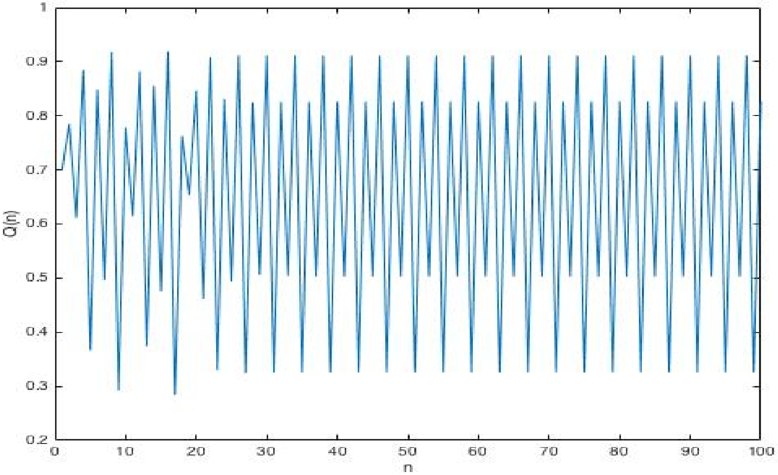
Back to the Question

- The coupled-logistic system has a stable out of phase 2-periodic orbit for intermediate values of migration, even for those values of growth rates that a single logistic system would exhibit chaos for.
- This is a system that has multiple attractors, and only analysing two of them means that we can't speak of the global picture because the dynamics highly depends on initial conditions.
- Now, **What sort of dynamics is preferable for a population as a whole to persist?**
    - Chaotic? Possibility of extinction is present.
    - Periodic Orbit will not oscillate to extinction.

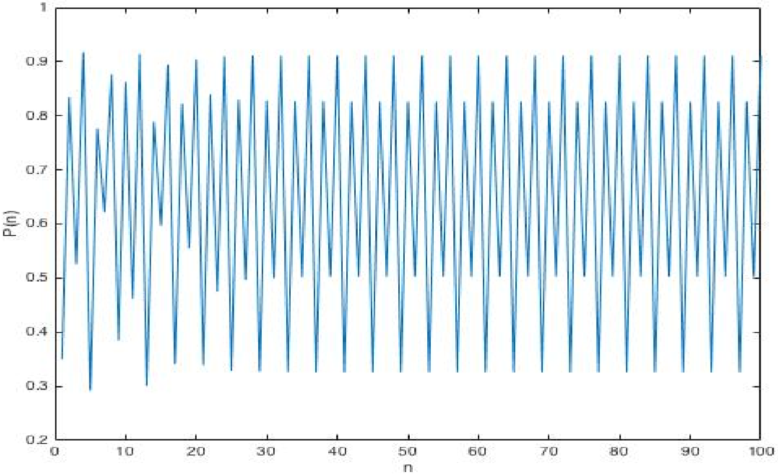
The Answer: Stabilisation of Local Population Dynamics

- For regions in parameter space, both chaotic and periodic solutions may exist. However, in the basin of attraction of the periodic solution, we can be sure that the population is stabilised in the sense that it won't go to extinction. We have only discussed the 2-periodic solution because it can be easily done but the general idea can be extended to n-periodic orbits. In this way, migration stabilises local population dynamics.
- Two plots are given here, one for r=3.6, D=0.075 and one for r=3.7, D=0.12.

r=3.7, D=0.12, X



r=3.7, D=0.12, Y



Conclusion

- While we have only studied this for the coupled logistic system, one of the **many possible meta-population models**, the fact that in such patch models, **dispersal (or migration) can be stabilizing is quite a general phenomena.**
  - In **insect populations**, local movement in a patchy environment can help otherwise unstable host and parasitoid populations to persist together.[4]
  - A **predator-prey interaction** between spatially dispersed populations can be considered as many local interactions connected by dispersal. Even when the local subsystems quickly become extinct in isolation, an ensemble of interconnected cells can, under certain conditions, persist much longer. [5]

[4] Hassel, M.P., Comins, H.N., May R.M. (1991) Spatial structure and chaos in insect population dynamics

[5] Crowley, P.H. Dispersal and the stability of predator-prey interactions